# ENERGY AND STABILITY OF PAIS-UHLENBECK OSCILLATOR

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Abstract. We study stability of higher-derivative dynamics from the viewpoint of more general correspondence between symmetries and conservation laws established by the Lagrange anchor. We show that classical and quantum stability may be provided if a higher-derivative model admits a bounded from below integral of motion and the Lagrange anchor that relates this integral to the time translation.

## Introduction

A notorious trouble appears when the Noether theorem [1] is applied to the theories with higher derivatives, the models whose Lagrangians depend on accelerations and higher derivatives of generalized coordinates. In contrast to the-lower order theories, where unboundedness of the canonical energy from below usually indicates the presence of ghost states and instability of the model, in higher-derivative theories the unboundedness of canonical energy is not necessary to have negative impact on classical dynamics [2, 3]. The relationship between (un)boundedness of canonical energy from below and (in)stability of higher-derivative theory was subject of many works [4–8].

In this note, we consider a stability of higher-derivative dynamics from the viewpoint of more general correspondence between symmetries and conservation laws which is established by the Lagrange anchor. Following the ideas of [9, 10], we show that the stability of higher-derivative theory may be provided if the model admits a bounded from below integral of motion and a Lagrange anchor that associates the integral of motion with translation in time. The general construction is illustrated by the example of the Pais-Uhlenbeck oscillator.

The paper is organized as follows. In Section 1, we recall some basic facts about the conservation laws of the Pais-Uhlenbeck (PU) oscillator. In Section 2, we introduce the Lagrange anchor for the PU oscillator and establish a correspondence between symmetries and conservation laws. The bounded integral of motion and the Lagrange anchor that associates it to time translations are identified. Section 3 is devoted to the Pais-Uhlenbeck oscillator with equal

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frequencies. We show that in unstable theory the bounded integral of motion exists, but it appears to be unrelated to the time-translation symmetry.

## 1. Conservation laws of the PU oscillator

We consider the one-dimensional Pais-Uhlenbeck oscillator of order $2n$ [11], whose action functional has the form

$$S[x(t)]=\int dtL\,,\qquad L=\frac{1}{2\Omega}x(t)\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)x(t)\,;\tag{1}$$

here

$$0<\omega_1<\omega_2<\ldots<\omega_n$$

are the frequencies of oscillations. We assume that there is no resonance, so that all the frequencies are different. We also introduced the dimensional constant $\Omega>0$ to provide the correct dimension of the action (1).

The corresponding equation of motion reads

$$\frac{\delta S}{\delta x}\equiv\frac{1}{\Omega}\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)x=0\,.\tag{2}$$

It is convenient to introduce the wave operator that defines r.h.s. of the equation of motion

$$T=\frac{1}{\Omega}\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)\,,\qquad\frac{\delta S}{\delta x}=T(x)\,.\tag{3}$$

We will also use notation

$$\dot{x}=\frac{dx}{dt},\qquad\ddot{x}=\frac{d^2x}{dt^2},\qquad\ldots,\qquad\overset{(n)}{x}=\frac{d^nx}{dt^n}$$

for the time derivatives of $x$.

The general solution of equation (2) is given by the sum of $n$ oscillations with frequencies $\omega_i$ with different amplitudes $A_i$ and phases $\varphi_i$

$$x(t)=\sum_{i=1}^{n}x_i(t)\,,\qquad x_i(t)\equiv A_i\sin(\omega_it+\varphi_i)=\mathcal{P}_ix(t)\,.\tag{4}$$

Here, the operators

$$\mathcal{P}_i=\prod_{j\neq i}\frac{1}{\omega_j^2-\omega_i^2}\Big(\frac{d^2}{dt^2}+\omega_j^2\Big)$$

have the sense of projectors to the subspaces of solutions with frequencies $\omega_i$, respectively. There are two obvious properties

$$\sum_{i=1}^{n}\mathcal{P}_i = 1\,,$$
$$\mathcal{P}_i^2 = \mathcal{P}_i - \Omega\sum_{j\neq i}\prod_{r\neq i}\frac{1}{\omega_i^2-\omega_r^2}\prod_{s\neq j}\frac{1}{\omega_j^2-\omega_s^2}\prod_{k\neq j,i}\Big(\frac{d^2}{dt^2}+\omega_k^2\Big)T\,. \tag{5}$$

In order to prove the first relation one can apply to it the Fourier transform:

$$F(\sum_{i=1}^{n}\mathcal{P}_i - 1) = \sum_{i=1}^{n}\prod_{j\neq i}\frac{1}{\omega_j^2-\omega_i^2}\Big(\omega_j^2-p^2\Big) - 1\,.$$

Then the r.h.s. of the relation is a polynomial of degree $2n-2$ that has $2n$ roots $p=\pm\omega_i$ and thus has to be equal to zero identically. The second relation follows from identity

$$\mathcal{P}_i^2 = \mathcal{P}_i\Big(1-\sum_{j\neq i}\mathcal{P}_j\Big)$$

with account of notation (3).

Due to relations (5), formula (4) establishes a correspondence between the solutions to the PU oscillator equation and the system of $n$ independent harmonic oscillators

$$\frac{\delta S}{\delta x(t)} = 0 \qquad \Leftrightarrow \qquad \Big(\frac{d^2}{dt^2}+\omega_i^2\Big)x_i(t) = 0, \qquad i=1,2,\ldots,n. \tag{6}$$

From (6) it immediately follows that the PU oscillator has $n$ independent integrals of motion

$$I_i \equiv \frac{1}{2}\Big[\dot{x}_i^2+\omega_i^2x_i^2\Big] = \frac{1}{2}\Big[\Big(\mathcal{P}_i\dot{x}\Big)^2+\omega_i^2\Big(\mathcal{P}_ix\Big)^2\Big]\,. \tag{7}$$

The general quadratic integral of motion is given by the linear combination of integrals (7). Namely,

$$I = -\sum_{i=1}^{n}\Big[\prod_{j\neq i}\Big(\omega_j^2-\omega_i^2\Big)\Big]\frac{\alpha_iI_i}{\Omega}\,, \tag{8}$$

with $\alpha_i$ being arbitrary real constants.

It is easy to see that expression $I$ in (8) is conserved

$$\frac{dI}{dt} = Q\frac{\delta S}{\delta x}\,, \qquad Q(t,x,\dot{x},\ldots,\overset{(2n-1)}{x}) = -\sum_{i=1}^{n}\Big(\alpha_i\mathcal{P}_i\dot{x}\Big)\,, \tag{9}$$

where the coefficient $Q$ is called the *characteristic* associated with the conservation law $I$. It is known [12] that there is a one-to-one correspondence between integrals of motion and characteristics. The last fact allows one to use characteristics for establishing a correspondence between the symmetries and conservation laws. The simplest example is provided by the

Noether theorem. The Noether theorem identifies the characteristic $Q$ with the infinitesimal symmetry transformation of action functional:

$$\delta_\varepsilon x = \varepsilon Q\,, \quad \delta_\varepsilon S = 0 \qquad \Leftrightarrow \qquad \frac{dI}{dt} = Q\frac{\delta S}{\delta x}\,. \tag{10}$$

The problem appears when a conservation law bounded form below is associated with the time translations. The general bounded conservation law (8) with $(-1)^i\alpha_i > 0$ corresponds to some symmetry of the action (1)

$$\delta_\varepsilon x = -\varepsilon\sum_{i=1}^{n}\Big(\alpha_i\mathcal{P}_i\dot{x}\Big)\,, \tag{11}$$

while the infinitesimal time translation $\delta_\varepsilon x = -\varepsilon\dot{x}$ corresponds to the unbounded conservation law with $\alpha_i = 1$. This is manifestation of general no-go statement about unboundedness of energy in the theories with higher derivatives. Unless the higher-derivative theory is highly constrained, the usual Noether theorem can't connect a positive conserved quantity to the time translation invariance, see for instance discussion in [8] and references therein.

## 2. The Lagrange anchor for the PU oscillator

The generalization of the Noether theorem suggests that the correspondence between the symmetries and characteristics (and hence conservation laws) is established by the linear differential operator[1]

$$V = \sum_{i=1}^{2n-1}\overset{(i)}{V}(t, x, \dot{x}, \ldots, \overset{(2n-1)}{x})\frac{d^i}{dt^i}$$

that associates the characteristic $Q$ with the symmetry

$$\delta_\varepsilon x = \varepsilon V(Q) = \sum_{i=1}^{2n-1}\overset{(i)}{V}\frac{d^iQ}{dt^i}\,, \qquad \delta_\varepsilon\Big(\frac{\delta S}{\delta x}\Big)\Big|_{\frac{\delta S}{\delta x}=0} = 0\,. \tag{12}$$

Invariance of the equation of motion under transformation (12) implies certain compatibility condition for the Lagrange anchor [12]. In the simplest case of linear equations $\delta S/\delta x = T(x)$ and the Lagrange anchor with no field and time dependence,

$$\overset{(i)}{V} = \text{const}, \qquad i = 0, \ldots, 2n-1\,,$$

this compatibility condition takes the form [9]

$$V^*T^* - TV = 0\,. \tag{13}$$

[1]The notation of this section is adapted for the case of the PU oscillator. For general definitions of the Lagrange anchor and correspondence between symmetries and conservation laws see [12, 13].

For a self-adjoint wave operator $T = T^*$ (which is always true for Lagrangian theories) and a self-adjoint Lagrange anchor $V = V^*$, relation (13) takes even more simple form

$$[V, T] = 0\,. \tag{14}$$

The obvious solution $V = 1$ corresponds to the canonical Lagrange anchor which is always admissible for Lagrangian theory. It establishes the Noether correspondence between symmetries and conservation laws (10). The canonical Lagrange anchor can't connect bounded from below integral of motion with translation in time, we have to be interested in the non-canonical Lagrange anchors. We have the following $n$-parameter family of the Lagrange anchors for the PU oscillator of order $2n$:

$$V = \sum_{i=1}^{n} \beta_i \mathcal{P}_i\,, \tag{15}$$

with $\beta_i$ being arbitrary real constants. The details about derivation of this Lagrange anchor can be found in [9].

The Lagrange anchor (15) associates the general conservation law (8) with the symmetry

$$\delta_\varepsilon x = \varepsilon V(Q) = -\varepsilon \sum_{i=1}^{n} \Big(\alpha_i \beta_i \mathcal{P}_i \dot{x}\Big) + \varepsilon \Omega R(\alpha_i, \beta_j) T(x)\,, \tag{16}$$

where

$$R(\alpha_i, \beta_j) = \sum_{i,j=1}^{n} \Big(\alpha_i \beta_i - \alpha_i \beta_j\Big) \prod_{r \neq i} \frac{1}{\omega_i^2 - \omega_r^2} \prod_{s \neq j} \frac{1}{\omega_j^2 - \omega_s^2} \prod_{k \neq j,i} \Big(\frac{d^2}{dt^2} + \omega_k^2\Big)\,.$$

The second term in (16) is given by the linear combination of equations of motion and their differential consequences, and thus should be considered as trivial. Below, we will consider symmetries modulo trivial terms.

The crucial difference between Noether's correspondence (10) and (16) is that the symmetry (16) depends on $n$ free parameters. We can use this ambiguity of the Lagrange anchor to connect the general integral of motion (8) with translations in time. Whenever $\alpha_i \neq 0$ the desired correspondence

$$V(Q) = -\dot{x} + \Omega R(\alpha_i, 1/\alpha_j) T(x) \tag{17}$$

is established for

$$\beta_i = \frac{1}{\alpha_i}\,. \tag{18}$$

In contrast to the Noether energy, the conservation law (8) can be bounded or unbounded from below depending on the value of $\alpha$'s. The bounded integrals of motion (8) are associated

with time translations by differential Lagrange anchor

$$V=\sum_{i=1}^{n}\frac{1}{\alpha_i}\mathcal{P}_i,\qquad (-1)^i\alpha_i>0\,. \tag{19}$$

From the classical viewpoint the relationship (12) is as good as the Noether's one. In particular, it allows one to define the generalization of the Dickey bracket of conservation laws and admits BRST-description [14]. Thus, the correspondence between the bounded from below conservation law (8), the Lagrange anchor (19) and the time translation (17) ensures the stability of the PU oscillator theory even if the canonical energy is unbounded.

Let us give one more argument that makes analogy between the energy and conservation law associated with the time translation more explicit. It is well known that different Lagrange anchors result in different quantizations of one and the same classical system [13, 15]. In the first-order formalism, the integrable[2] Lagrange anchor always defines a Poisson bracket on the phase space of the system, while the corresponding integral of energy becomes the Hamiltonian [13, 16, 17]. The canonical Lagrange anchor corresponds to the canonical Poisson brackets and Hamiltonian that follows from the Ostrogradsky formalism [18, 19], while non-canonical Lagrange anchors correspond to non-canonical Poisson brackets and Hamiltonians.

## 3. The case of resonance

Let us consider the 4-th order PU oscillator in the case of equal frequencies $\omega=\omega_1=\omega_2$. The equation of motion reads

$$T(x)=\Big(\frac{1}{\omega}\frac{d^2}{dt^2}+\omega\Big)^2x=0\,,\qquad T=\Big(\frac{1}{\omega}\frac{d^2}{dt^2}+\omega\Big)^2. \tag{20}$$

The solutions to the equation of motion demonstrate runaway behavior with linear time dependence of the oscillation amplitude

$$x(t)=A\sin(\omega t+\varphi_0)+Bt\sin(\omega t+\varphi_1)$$

with $A$, $B$, $\varphi_0$ and $\varphi_1$ being arbitrary real constants. The system (20) still has two impendent integrals of motion

$$I_1=\frac{1}{2}\Big(\frac{1}{\omega^2}\dddot{x}+\dot{x}\Big)^2+\frac{1}{2}\Big(\frac{1}{\omega}\ddot{x}+\omega x\Big)^2,\quad I_2=\frac{1}{2}\left(\frac{\ddot{x}^2-2\dddot{x}\dot{x}}{\omega^2}-2\dot{x}^2-\omega^2x^2\right).$$

The first integral is obtained from (7) by taking limit $\omega_1\to\omega_2$ with special renormalization of the $\alpha-$constants. The second one is just the Noether energy. In contrast to the case of unequal frequencies, it is impossible to find two independent bounded from below quadratic integrals

[2] See the definition of integrability in [12]. The field-independent Lagrange anchor (15) is automatically integrable.

of motion. Only the integral of motion $I_1$ is bounded from below. However, an attempt to associate it with time translation fails.

The characteristic for $I_1$ reads

$$\frac{dI_1}{dt} = Q_1 T(x), \qquad Q_1 = \left(\frac{1}{\omega^2}\dddot{x} + \dot{x}\right).$$

There are two-parameter family of Lagrange anchors for PU oscillator (20)

$$V = \frac{\beta_2}{\omega^2}\frac{d^2}{dt^2} + \beta_1 . \tag{21}$$

The corresponding symmetry reads

$$V(Q_1) = \frac{\beta_1}{\omega^2} T(x) + \frac{\beta_2 - \beta_1}{\omega^2}\dddot{x} + (\beta_2 - \beta_1)\dot{x} . \tag{22}$$

In (22), the third derivative vanishes if and only if the symmetry (22) is trivial, i.e., $\beta_1 = \beta_2$. In view of above, there is no time-independent bounded from below conservation law that could be associated to time translation. This result demonstrates the fact that has been already observed in [7], where it was found that PU oscillator with resonance does not admit Hamiltonian formulation with any bounded Hamiltonian.

## Conclusion

We observe that for higher-derivative theories, the stability does not necessarily require the Noether energy to be bounded from below. The classical stability can be ensured by a weaker condition that the model admits a bounded integral of motion. Once the equations of motion admit the Lagrange anchor such that maps the bounded integral to the time translation, the theory can retain stability at quantum level. Both the conserved quantity and the Lagrange anchor are not uniquely defined by the equations of motion and may exist even in non-singular models. This allows us to expand the stability analysis to the wide class of higher-derivative theories, including non-singular ones. The general idea is exemplified by the Pais-Uhlenbeck oscillator. Using the ambiguity of choice of the Lagrange anchor and bounded conserved quantity, we demonstrate the stability of Pais-Uhlenbeck oscillator when all the frequencies are different.

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